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Minimal Number of Operators for Observability of N-Level Quantum Systems[1]

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This paper discusses the minimal number $n_{\min}$ of operators $A_1, \ldots, A_n$, whose expectation values at some instants determine the statistical state of an N-level quantum system. We assume that the macroscopic information about the system in question is given by the mean values $\mathrm{Tr}[\rho(t_j)A_i] = m_i(t_j)$ of n self-adjoint operators $A_1, \ldots, A_n$ at some instants $t_1 < t_2 < \cdots < t_s$, where $s < N^2 - 1$.

1. INTRODUCTION

The notion of observability of a dynamical system is associated with the processing of data obtained from observation of the system. Usually one assumes (in the classical and quantum case) that our knowledge about the system in question is given (at the moment $t = t_0$) by the mean values $\mathrm{Av}(A_1), \ldots, \mathrm{Av}(A_n)$ of the observables (physical quantities) $\mathcal{A}_1, \ldots, \mathcal{A}_n$, e.g., in quantum case, by the equalities

$$\mathrm{Av}(A_i) = \mathrm{Tr}[\rho(t_0)A_i] = m_i \tag{1}$$

for $i = 1, \ldots, n$. In the above equalities $\rho(t_0)$ denotes a density operator which describes the state of the system in question at the moment t_0. In general, to determine the state $\rho(t_0)$ from the equalities (1) we have to use information theory methods, i.e., the so-called "principle of maximum entropy" (Jaynes, 1957; Ingarden and Urbanik, 1962; cf. also Katz, 1967; and Wichman, 1963).

The purpose of this paper is to study the case when our knowledge about the system in question is given by the conditions of the form

$$\mathrm{Av}(A_i)(t_j) = \mathrm{Tr}(\rho(t_j)A_i) = m_i(t_j) \tag{2}$$

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where $\rho(t_j)$ describes the state of the system at the moment t_j and $t_0 \leqslant t_1 < \cdots < t_s \leqslant t_0 + T$ for $T > 0$. Here the symbols A_i for $i = 1, \ldots, n$ denote self-adjoint operators which córrespond to physical quantities $\mathcal{C}_1, \ldots, \mathcal{C}_n$. In other words, we assume that from observations on the system we are able to known the assignments

$$[t_0, t_0 + T] \ni t_j \mapsto m_i(t_j) \in \mathbb{R}^1 \tag{3}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, s$. We know that with each measurement of an observable we have associated a change of state of the system (cf., e.g., Davies, 1976; Jauch, 1968). In what follows we consider $\nu = ns$ ensembles of identical systems (systems prepared by identical procedures) and we assume that at $t = t_0$ all ensembles are described by the same density operator $\rho(t_0)$. The quantities $m_i(t_j)$, where $i = 1, \ldots, n$ and $j = 1, \ldots, s$ can be then measured independently—one can measure only one quantity in each ensemble.

The time evolution of the system in question is governed by the von Neumann equation

$$i\dot{\rho}(t) = [H, \rho(t)] \tag{4}$$

where H denotes the Hamiltonian of the system and we assume that the Planck constant $\hbar = 1$. In other words, the dynamics of such system is given by a continuous, one-parameter group of unitary transformations

$$\rho(t) = U(t, t_0)\rho(t_0)U^*(t, t_0) \tag{5}$$

where $U(t, t_0) := \exp[-iH(t - t_0)]$ and $t \in \mathbb{R}^1$. The problem of observability consist in the establishment of a current state $\rho(t)$ [or equivalently initial state $\rho(t_0)$] of the system by known assignments (3) and the equation (4).

Let $\mathbb{S}$ be an isolated N-level quantum system which is characterized by the Hamiltonian H. The following question may arise: what is the minimal number $n_{\min}$ of self-adjoint operators $A_1, \ldots, A_n$ for which the system $\mathbb{S}$ can be uniquely described by information of the form (2)? To give an exact answer to this question we must consider the notion of observability of dynamical systems in detail.

2. OBSERVABILITY OF N-LEVEL QUANTUM SYSTEMS

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space associated with some quantum system $\mathbb{S}$. Let $\dim \mathcal{H} = N$. We will denote by $B(\mathcal{H})$ N^2-dimensional vector space of all linear operators on $\mathcal{H}$. One can consider the space $B(\mathcal{H})$ as a Hilbert space if we define scalar product of the operators

$A, B \in B(\mathcal{H})$ by the equality

$$((A, B)) := \mathrm{Tr}(A^* B) \tag{6}$$

The set of all states of the system $\mathbb{S}$

$$\mathcal{P}(\mathcal{H}) := \{\rho \in B(\mathcal{H});\ \rho = \rho^*, \rho \geqslant 0, \mathrm{Tr}\,\rho = 1\} \tag{7}$$

is a proper subset of the space $B(\mathcal{H})$.

We assume that the time evolution of the system $\mathbb{S}$ is described by the equation

$$\dot{\rho}(t) = L\rho(t) \tag{8}$$

where L: $B(\mathcal{H}) \to B(\mathcal{H})$ is linear operator given by

$$L: \rho \mapsto L\rho = -i[H, \rho] \tag{9}$$

In order to simplify the notation we will denote $\rho(t_0)$ by ρ_0. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a set of n self-adjoint linearly independent operators ("observables" in physical terminology).

Using terminology from control theory (cf., e.g., Kalman, 1969) we formulate the following definitions:

Definition 1. The quantum state $\rho_0 \in \mathcal{P}(\mathcal{H})$ is called $\mathbf{A}$-observable if for every $T > 0$ there exists at least one set of instants $t_1, \ldots, t_s$, where $t_0 \leqslant t_1 < \cdots < t_s \leqslant t_0 + T$, such that ρ_0 can be reconstructed using the assignments

$$[t_0, t_0 + T] \ni t_j \mapsto m_i(t_j) = \mathrm{Tr}[A_i \rho(t_j)] \tag{10}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, s$.

Definition 2. The quantum system $\mathbb{S}$ is said to be $\mathbf{A}$-observable if all states $\rho \in \mathcal{P}(\mathcal{H})$ are $\mathbf{A}$-observable.

Now we want to formulate necessary and sufficient conditions for observability of quantum systems in terms of the Hamiltonian H and operators $A_1, \ldots, A_n$. For this, we introduce the following notation. If $A_j \in B(\mathcal{H})$ for $j = 1, \ldots, n$ and H is a Hamiltonian of the system in question, then

$$\langle H(k) | A_j \rangle := \mathrm{Span}(Q_1^j, \ldots, Q_k^j) \tag{11}$$

where $Q_1^j := A_j$ for $j = 1, \ldots, n$ and the operators $Q_2^j, \ldots, Q_{N^2-1}^j$ are defined in the following way:

$$Q^j_{k+1} := i[H, Q^j_k] \tag{12}$$

for $j=1,\ldots,n$ and $k=1,\ldots,N^2-2$.

If we have two subspaces $\langle H(k_1)|A_1\rangle \subset B(\mathcal{H})$ and $\langle H(k_2)|A_2\rangle \subset B(\mathcal{H})$ then the algebraic sum of these subspaces is given by

$$\langle H(k_1)|A_1\rangle + \langle H(k_2)|A_2\rangle = \{A+B;\, A\in\langle H(k_1)|A_1\rangle,\, B\in\langle H(k_2)|A_2\rangle\} \tag{13}$$

This definition can be extended in the obvious way to finite collections of subspaces. It is well to note that $\langle H(k_1)|A_1\rangle + \langle H(k_2)|A_2\rangle$ is the span of $\langle H(k_1)|A_1\rangle$ and $\langle H(k_2)|A_2\rangle$ and may be much larger then the set-theoretic union.

We can now formulate our main results.

Theorem 1. Let $\mathbb{S}$ be an N-level quantum system, whose time evolution is described by von Neumann equation

$$\dot{\rho}(t) = L\rho(t) \tag{14}$$

where

$$L\rho = -i[H, \rho] \tag{14a}$$

for $\rho \in \mathcal{P}(\mathcal{H})$. Let the information about the system in question be given by assignments of the form

$$[t_0, t_0+T] \ni t_j \mapsto m_i(t_j) = \mathrm{Tr}[A_i \rho(t_j)] \tag{15}$$

for $i=1,\ldots,n$ and $j=1,\ldots,s$. The system $\mathbb{S}$ is **A**-observable if and only if

$$\langle H(m)|I\rangle + \sum_{i=1}^{n} \langle H(m)|A_i\rangle = B(\mathcal{H}) \tag{16}$$

where I denotes the identity in $B(\mathcal{H})$ and m denotes a degree of the minimal polynomial of the operator $L(\cdot) = -i[H, \cdot]$.

Proof. First of all we should note that if the quantum system $\mathbb{S}$ is **A**-observable, then the exact value of $\rho(t_0)$ is determinable from the measurement of $m_i(t)$ over a time duration $[t_0, t_0+T]$. Since

$$\rho(t) = \exp(Lt)\rho(t_0) \tag{17}$$

and $\exp(Lt)$ can be expressed as

$$\exp(Lt) = \sum_{k=0}^{m-1} \alpha_k(t) L^k \tag{18}$$

where m is a degree of the minimal polynomial of the operator $L(\cdot) = -i[H, \cdot]$, so we obtain

$$\rho(t) = \sum_{k=0}^{m-1} \alpha_k(t) L^k \rho(t_0) \tag{19}$$

The expression $m_i(t)$ can be now written as

$$\begin{aligned} m_i(t) &= \mathrm{Tr}[A_i \rho(t)] = ((A_i, \rho(t))) \\ &= \sum_{k=0}^{m-1} \alpha_k(t) ((A_i, L^k \rho(t_0))) \\ &= \sum_{k=0}^{m-1} \alpha_k(t) \left(((L^*)^k A_i, \rho(t_0)) \right) \end{aligned} \tag{20}$$

where L^* is the adjoint operator to L with respect to the inner product $((\ ,\ ))$.

Since the functions $\alpha_k(t)$ are linearly independent of each other (cf., e.g., Zadeh, 1963), then for every $T > 0$ there exists at least one set of instants $t_1, \ldots, t_m$ (m is a degree of the minimal polynomial of L), where $t_0 \leqslant t_1 < \cdots < t_m \leqslant t_0 + T$, such that

$$m_i(t_j) = \sum_{k=0}^{m-1} \alpha_k(t_j) \left(((L^*)^k A_i, \rho_0) \right) \tag{21}$$

and $\det[\alpha_k(t_j)] \neq 0$. Hence the above equations can be solved for $(((L^*)^k A_i, \rho_0))$, where $k = 0, 1, \ldots, m-1$, $i = 1, \ldots, n$ giving

$$\left(((L^*)^k A_i, \rho_0) \right) = C_k^i \tag{22}$$

Since $\rho_0 \in \mathcal{P}(\mathcal{H})$ we have also the condition

$$\operatorname{Tr} \rho_0 = ((I, \rho_0)) = 1 \tag{23}$$

Now, it is easy to see that from (22) and (23) one can determine the state ρ_0 if and only if the operators $I, A_i, L^*A_i, \ldots, (L^*)^{m-1}A_i$ (where $i = 1, \ldots, n$) span the vector space $B(\mathcal{H})$. According to the duality relations

$$\operatorname{Tr}[A_i(L\rho)] = \operatorname{Tr}[(L^*A_i)\rho] \tag{24}$$

we can check that in the Heisenberg picture

$$L^*A_1 = i[H, A_1] \tag{25}$$

for all $A_1 \in B(\mathcal{H})$. Then, using the notation given by (12) and (11), we see that our system $\mathbb{S}$ is A-observable if and only if

$$\langle H(m)|I\rangle + \sum_{i=1}^{n} \langle H(m)|A_i\rangle = B(\mathcal{H}) \tag{26}$$

This completes the proof.

3. THE MINIMAL NUMBER OF OPERATORS FOR OBSERVABILITY

In Theorem 1 we have formulated the conditions for observability of N-level quantum system with a given Hamiltonian H and given observables $A_1, \ldots, A_n$. The following question may arise: what is the minimal number $n_{\min}$ of self-adjoint operators $A_1, \ldots, A_n$ for which the system $\mathbb{S}$ is $(A_1, \ldots, A_n)$-observable. To give an exact answer on this question we must determine the number of nontrivial invariant factors of the operator L or equivalently, we must determine $\dim \operatorname{Ker}(L)$, where $L(\cdot) = -i[H, \cdot]$. In what follows we will discuss this problem in detail. Let us suppose that the spectrum of H is

$$\sigma(H) = \{\lambda_1, \ldots, \lambda_r\} \tag{27}$$

where $\lambda_1, \ldots, \lambda_r$ are all different eigenvalues of H. In other words we assume that if $i \neq j$ then $\lambda_i \neq \lambda_j$ for all $i, j = 1, \ldots, r \leqslant N$. Of course, we may also assume that $\lambda_1 < \lambda_2 < \cdots < \lambda_r$.

Now we can formulate the following theorem:

Theorem 2. Let $\mathbb{S}$ be an isolated N-level quantum system characterized by Hamiltonian H. The minimal number of observables $A_1,\ldots,A_n$ for which the system $\mathbb{S}$ can be $(A_1,\ldots,A_n)$-observable is given by the equality

$$n_{\min}=n_1^2+n_2^2+\cdots+n_r^2 \tag{28}$$

where

$$n_i=\dim\operatorname{Ker}(\lambda_i I-H) \tag{29}$$

for all $\lambda_i\in\sigma(H)$, $i=1,\ldots,r$.

Proof. First of all, let us observe that if $\sigma(H)=\{\lambda_1,\ldots,\lambda_r\}$, then the spectrum of the operator L is

$$\sigma(L)=\left\{\mu\in\mathbb{R}^1;\ \mu=\lambda_i-\lambda_j,\ \lambda_i,\lambda_j\in\sigma(H)\right\} \tag{30}$$

(cf., e.g., Spohn, 1976). Now, let us denote by n_i the so-called algebraic multiplicity of $\lambda_i\in\sigma(H)$, i.e., the multiplicity of λ_i as a root of the characteristic polynomial of H. Because H is self-adjoint operator therefore the algebraic multiplicity of λ_i is equal to the geometric multiplicity of λ_i, i.e., we can write

$$n_i=\dim\operatorname{Ker}(\lambda_i I-H) \tag{31}$$

for $i=1,\ldots,r$. Of course, we have also $\sum_{i=1}^r n_i=N=\dim\mathcal{H}$.

Using (30) we can see that the multiplicities of all eigenvalues of the operator L are not determined uniquely by multiplicities of $\lambda_i\in\sigma(H)$. But, if we suppose that $\lambda_1<\lambda_2<\cdots<\lambda_r$ then the multiplicity of the eigenvalue $\mu=\lambda_i-\lambda_j$, where $1\leqslant i<j\leqslant r$, is limited in the following way:

$$\nu=\dim\operatorname{Ker}\left[(\lambda_i-\lambda_j)I-L\right]\leqslant\sum_{\substack{k=1\\ l=2\\ (k<l)}}^{r} n_k n_l \tag{32}$$

It is easy to see that the minimal number of observables for which the system $\mathbb{S}$ is $(A_1,\ldots,A_n)$-observable is determined by the equality

$$n_{\min}=\max_{\substack{i=1,\ldots,r\\ j=1,\ldots,r}}\dim\operatorname{Ker}\left[(\lambda_i-\lambda_j)I-L\right] \tag{33}$$

Because for all natural numbers $n_1,\ldots,n_r$ we have the inequality

$$\sum_{\substack{k=1\\l=2\\(k<l)}}^{r} n_k n_l \leqslant n_1^2+n_2^2+\cdots+n_r^2 \tag{34}$$

then we can write [using inequality (32) and similar inequality for $i>j$]

$$n_{\min}=\dim\operatorname{Ker}L=n_1^2+n_2^2+\cdots+n_r^2 \tag{35}$$

This completes the proof. ■

Remark. In particular, when all subspaces $\operatorname{Ker}(\lambda_i I-H)$ are one-dimensional ($i=1,\ldots,N$), or in other words, when all eigenvalues of H are different from each other, we get from (28) that $n=N$, i.e., the number $n_{\min}$ is equal to the dimension of the space $\mathcal{H}$.

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